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Some properties of the mapping T_μ introduced by a representation in Banach and locally convex spaces

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Abstract. Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a semigroup S . We show that the mapping T_μ introduced by a mean on a subspace of $l^\infty(S)$ inherits some properties of \mathcal{S} in Banach spaces and locally convex spaces. The notions of Q - G -nonexpansive mapping and Q - G -attractive point in locally convex spaces are introduced. We prove that T_μ is a Q - G -nonexpansive mapping when T_s is Q - G -nonexpansive mapping for each $s \in S$ and a point in a locally convex space is Q - G -attractive point of T_μ if it is a Q - G -attractive point of \mathcal{S} .

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Introduction and preliminaries

Let C be a nonempty closed and convex subset of a Banach space E and E^* be the dual space of E . Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^* . The normalized duality mapping $J : E \rightarrow E^*$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\},$$

for all $x \in E$. For more details, see [13].

The space of all bounded real-valued functions defined on S with supremum norm is denoted by $l^\infty(S)$.

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l_s and r_s in $l^\infty(S)$ are defined as follows: $(l_t g)(s) = g(ts)$ and $(r_t g)(s) = g(st)$, for all $s \in S$, $t \in S$ and $g \in l^\infty(S)$.

Suppose that X is a (linear) subspace of $l^\infty(S)$ containing 1 and let X^* be its topological dual space. An element m of X^* is said to be a mean on X , provided $\|m\| = m(1) = 1$. For $m \in X^*$ and $g \in X$, $m_t(g(t))$ is often written instead of $m(g)$. Suppose that X is left invariant (respectively, right invariant), i.e., $l_t(X) \subset X$ (respectively, $r_t(X) \subset X$) for each $t \in S$. A mean m on X is called left invariant (respectively, right invariant), provided $m(l_t g) = m(g)$ (respectively, $m(r_t g) = m(g)$) for each $t \in S$ and $g \in X$. X is called left (respectively, right) amenable if X possesses a left (respectively, right) invariant mean. X is amenable, provided X is both left and right amenable.

Let D be a directed set in X . A net $\{m_\alpha : \alpha \in D\}$ of means on X is called left regular, provided

$$\lim_{\alpha \in D} \|l_t^* m_\alpha - m_\alpha\| = 0,$$

for every $t \in S$, where l_t^* is the adjoint operator of l_t .

Let E a reflexive Banach space. Let g be a function on S into E such that the weak closure of $\{g(s) : s \in S\}$ is weakly compact and suppose that X is a subspace of $l^\infty(S)$ owning all the functions $s \rightarrow \langle g(s), x^* \rangle$ with $x^* \in E^*$. We know from [3] that, for any $m \in X^*$, there exists a unique element g_m in E such that $\langle g_m, x^* \rangle = m_s \langle g(s), x^* \rangle$ for all $x^* \in E^*$. We denote such g_m by $\int g(s)m(s)$. Moreover, if m is a mean on X , then from [5], $\int g(s)m(s) \in \overline{\text{co}}\{g(s) : s \in S\}$, where $\overline{\text{co}}\{g(s) : s \in S\}$ denotes the closure of the convex hull of $\{g(s) : s \in S\}$.

The following definitions and basic results are needed in the next section.

- (1) Let E be a Banach space or a locally convex space, C be a nonempty closed and convex subset of E and S be a semigroup. Then, a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings from C into itself is called a representation of S as mappings on C into itself provided $T_{st}x = T_s T_t x$ for all $s, t \in S$ and $x \in C$. Note that, $\text{Fix}(\mathcal{S})$ is the set of common fixed points of \mathcal{S} , that is

$$\text{Fix}(\mathcal{S}) = \bigcap_{s \in S} \{x \in C : T_s x = x\}.$$

- (2) Let E be a real Banach space and C be a subset of E . The mapping $T : C \rightarrow C$ is called:

- a. nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
- b. quasi nonexpansive [10] if $\|Tx - f\| \leq \|x - f\|$ for all $x \in C$ and $f \in \text{Fix}(T)$, the fixed point set of T ;

- c. strongly quasi nonexpansive [10] if $\|Tx - f\| \leq \|x - f\|$ for all $x \in C \setminus \text{Fix}(T)$ and $f \in \text{Fix}(T)$;
- d. F -quasi nonexpansive (for a subset $F \subseteq \text{Fix}(T)$) if $\|Tx - f\| \leq \|x - f\|$ for all $x \in C$ and $f \in F$;
- e. strongly F -quasi nonexpansive [10] (for a subset $F \subseteq \text{Fix}(T)$) if

$$\|Tx - f\| \leq \|x - f\|,$$

for all $x \in C \setminus \text{Fix}(T)$ and $f \in F$,

- f. retraction [10] if $T^2 = T$,
- g. asymptotically nonexpansive [6] if for all $x, y \in C$ the following inequality holds:

$$\limsup_{n \rightarrow \infty} \|T^n x - T^n y\| \leq \|x - y\|. \quad (0.1)$$

- (3) Suppose that $\mathcal{S} = \{T_s : s \in S\}$ is a representation of a semigroup S on a set C in a Banach space E . An element $a \in E$ is called:

- a. asymptotically attractive point of S for C provided

$$\limsup_{n \rightarrow \infty} \|a - T_t^n x\| \leq \|a - x\|, \quad (0.2)$$

for all $t \in S$ and $x \in C$,

- b. uniformly asymptotically nonexpansive representation, if for each $x, y \in C$,

$$\limsup_{n \rightarrow \infty} \sup_t \|T_t^n x - T_t^n y\| \leq \|x - y\|, \quad (0.3)$$

- c. uniformly asymptotically attractive point, if for each $x \in C$,

$$\limsup_{n \rightarrow \infty} \sup_t \|a - T_t^n x\| \leq \|a - x\|. \quad (0.4)$$

- (4) Let X be a locally convex topological vector space (for short, locally convex space) generated by a family of seminorms Q , C be a nonempty closed and convex subset of X and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ (for more details refer to [4]). A mapping T of C into itself is called Q - G -nonexpansive if $q(Tx - Ty) \leq q(x - y)$, whenever $(x, y) \in E(G)$ for any $x, y \in C$ and $q \in Q$, and a mapping f is a Q -contraction on E if $q(f(x) - f(y)) \leq \beta q(x - y)$, for all $x, y \in E$ such that $0 \leq \beta < 1$.

It is easy to see that the locally convex space X generated by a family of seminorms Q is separated (Hausdorff) if and only if the family of seminorms Q possesses the following property:

for each $x \in X \setminus \{0\}$ there exists $q \in Q$ such that $q(x) \neq 0$ or equivalently

$$\bigcap_{q \in Q} \{x \in X : q(x) = 0\} = \{0\},$$

(see [1]).

The following results play crucial role in the next section.

Lemma 1. [12, 3] *Suppose that g is a function of S into E such that the weak closure of $\{g(t) : t \in S\}$ is weakly compact and let X be a subspace of $l^\infty(S)$ containing all the functions $t \rightarrow \langle g(t), x^* \rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element g_μ in E such that*

$$\langle g_\mu, x^* \rangle = \mu_t \langle g(t), x^* \rangle,$$

for all $x^* \in E^*$. Moreover, if μ is a mean on X then

$$\int g(t) d\mu(t) \in \overline{\text{co}} \{g(t) : t \in S\}.$$

We can write g_μ by

$$\int g(t) d\mu(t).$$

Next, we will need some concepts in locally convex spaces.

Consider a family of seminorms Q on the locally convex space X which determines the topology of X and the seminorm $q \in Q$. Let Y be a subset of X , we put

$$q_Y^*(f) = \sup\{|f(y)| : y \in Y, q(y) \leq 1\}$$

and

$$q^*(f) = \sup\{|f(x)| : x \in X, q(x) \leq 1\},$$

for every linear functional f on X . Observe that, for each $x \in X$ that $q(x) \neq 0$ and $f \in X^*$, then $|\langle x, f \rangle| \leq q(x)q^*(f)$. We will make use of the following Theorems.

Theorem 1. [2] *Suppose that Q is a family of seminorms on a real locally convex space X which determines the topology of X and $q \in Q$ is a continuous seminorm and Y is a vector subspace of X such that*

$$Y \cap \{x \in X : q(x) = 0\} = \{0\}.$$

Let f be a real linear functional on Y such that $q_Y^(f) < \infty$. Then there exists a continuous linear functional h on X that extends f such that $q_Y^*(f) = q^*(h)$.*

Theorem 2. [2] Suppose that Q is a family of seminorms on a real locally convex space X which determines the topology of X and $q \in Q$ a nonzero continuous seminorm. Let x_0 be a point in X . Then there exists a continuous linear functional f on X such that $q^*(f) = 1$ and $f(x_0) = q(x_0)$.

Consider a reflexive Banach space E , a nonempty closed convex subset C of E , a semigroup S and a representation $\mathcal{S} = \{T_s : s \in S\}$ of S and let X be a subspace of $l^\infty(S)$ and μ be a mean on X . We write $T_\mu x$ instead of $\int T_t x d\mu(t)$. The relations between the representation \mathcal{S} and the mapping T_μ have been studied by many authors, for instance see [6, 7, 10, 11].

In this paper, we establish some relations between the representation \mathcal{S} and T_μ in Banach and locally convex spaces.

1 Main results

In the following theorem, we prove that T_μ inherits some properties of representation \mathcal{S} in Banach spaces.

Theorem 3. Suppose that C is a nonempty closed, convex subset of a reflexive Banach space E , S a semigroup, $\mathcal{S} = \{T_s : s \in S\}$ a representation of S as self mappings on C such that weak closure of $\{T_t x : t \in S\}$ is weakly compact for each $x \in C$. If X is a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \rightarrow \langle T_t x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$, then the following assertions hold:

- (a) Let the mapping $t \rightarrow \langle T_t^n x - T_t^n y, x^* \rangle$ be an element of X for each $x, y \in C$, $n \in \mathbb{N}$ and $x^* \in E$. Let μ be a mean on X and $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as uniformly asymptotically nonexpansive self mappings on C , then T_μ is an asymptotically nonexpansive self mapping on C ,
- (b) $T_\mu x = x$ for each $x \in \text{Fix}(\mathcal{S})$,
- (c) $T_\mu x \in \overline{\text{co}}\{T_t x : t \in S\}$ for each $x \in C$,
- (d) if X is r_s -invariant for each $s \in S$ and μ is right invariant, then $T_\mu T_t = T_\mu$ for each $t \in S$,
- (e) let $a \in C$ be a uniformly asymptotically attractive point of \mathcal{S} and the mapping $t \rightarrow \langle a - T_t^n x, x^* \rangle$ be an element of X for each $x \in C$, $n \in \mathbb{N}$ and $x^* \in E$. Then a is an asymptotically attractive point of T_μ ,
- (f) let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as the affine self mappings on C , then T_μ is an affine self mapping on C ,

- (g) let P be a self mappings on C that commutes with $T_s \in \mathcal{S} = \{T_s : s \in S\}$ for each $s \in S$. Let the mapping $t \rightarrow \langle PT_t x, x^* \rangle$ be an element of X for each $x \in C$ and $x^* \in E$. Then T_μ commutes with P ,
- (h) let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as quasi nonexpansive self mappings on C , then T_μ is a $\text{Fix}(\mathcal{S})$ -quasi nonexpansive self mapping on C ,
- (i) let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as F -quasi nonexpansive self mappings on C (for a subset $F \subseteq \text{Fix}(\mathcal{S})$), then T_μ is an F -quasi nonexpansive self mapping on C ,
- (j) let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as strongly F -quasi nonexpansive self mappings on C (for a subset $F \subseteq \text{Fix}(\mathcal{S})$), then T_μ is an strongly F -quasi nonexpansive self mapping on C ,
- (k) let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as retraction self mappings on C , then T_μ is a retraction self mapping on C ,
- (l) let $E = H$ be a Hilbert space and $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as monotone self mappings on H , then T_μ is a monotone self mapping on H .

Proof. (a) Since \mathcal{S} is a representation as uniformly asymptotically nonexpansive self mappings on C , hence, from (0.3) and the part (b) of Theorem 3. 1. 7 in [8], there exists an integer $m_0 \in \mathbb{N}$ such that

$$\sup_t \|T_t^n x - T_t^n y\| \leq \|x - y\|,$$

for all $n \geq m_0$, $x, y \in C$. Suppose that $x_1^* \in J(T_\mu^n x - T_\mu^n y)$ and $x, y \in C$, where J is the normalized duality mapping on E . We know from [3], see Lemma 1.1, that for any $\mu \in X^*$, there exists a unique element f_μ in E such that

$$\langle f_\mu, x^* \rangle = \mu_s \langle f(s), x^* \rangle, \quad (1.1)$$

for all $x^* \in E^*$, where f is a function of S into E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact. Then from (1.1) we have

$$\begin{aligned} \|T_\mu^n x - T_\mu^n y\|^2 &= \langle T_\mu^n x - T_\mu^n y, x_1^* \rangle = \mu_t \langle T_t^n x - T_t^n y, x_1^* \rangle \\ &\leq \sup_t \|T_t^n x - T_t^n y\| \|T_\mu^n x - T_\mu^n y\| \\ &\leq \|x - y\| \|T_\mu^n x - T_\mu^n y\|, \end{aligned}$$

and

$$\|T_\mu^n x - T_\mu^n y\| \leq \|x - y\|,$$

for all $n \geq m_0$, $x, y \in C$. Therefore, we get

$$\limsup_{n \rightarrow \infty} \|T_\mu^n x - T_\mu^n y\| \leq \|x - y\|.$$

(b) Suppose that $x \in \text{Fix}(\mathcal{S})$ and $x^* \in E^*$. Hence

$$\langle T_\mu x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \mu_t \langle x, x^* \rangle = \langle x, x^* \rangle.$$

(c) The assertion follows from Lemma 1.

(d) It follows from

$$\langle T_\mu(T_s x), x^* \rangle = \mu_t \langle T_{ts} x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \langle T_\mu x, x^* \rangle.$$

(e) Since a is a uniformly attractive point, hence, from (0.4) and from part (b) of Theorem 3. 1. 7 in [8], for each $x \in C$ there exists an integer $m_0 \in \mathbb{N}$ such that

$$\sup_t \|a - T_t^n x\| \leq \|a - x\|,$$

for all $n \geq m_0$. Suppose that $x_2^* \in J(a - T_\mu^n x)$, therefore from (1.1) we have,

$$\begin{aligned} \|a - T_\mu^n x\|^2 &= \langle a - T_\mu^n x, x_2^* \rangle = \mu_t \langle a - T_t^n x, x_2^* \rangle \\ &\leq \sup_t \|a - T_t^n x\| \|a - T_\mu^n x\| \\ &\leq \|a - x\| \|a - T_\mu^n x\|. \end{aligned}$$

Hence,

$$\|a - T_\mu^n x\| \leq \|a - x\|,$$

for all $n \geq m_0$. Thus, we get

$$\limsup_{n \rightarrow \infty} \|a - T_\mu^n x\| \leq \|a - x\|,$$

for each $x \in C$.

(f) If $x_1^* \in E^*$, then for all positive integers α, β and $x, y \in C$ with $\alpha + \beta = 1$, we have

$$\begin{aligned} \langle T_\mu(\alpha x + \beta y), x_1^* \rangle &= \mu_t \langle T_t(\alpha x + \beta y), x_1^* \rangle \\ &= \mu_t \langle \alpha T_t x + \beta T_t y, x_1^* \rangle \\ &= \alpha \mu_t \langle T_t x, x_1^* \rangle + \beta \mu_t \langle T_t y, x_1^* \rangle \\ &= \alpha \langle T_\mu x, x_1^* \rangle + \beta \langle T_\mu y, x_1^* \rangle \\ &= \langle \alpha T_\mu x + \beta T_\mu y, x_1^* \rangle, \end{aligned}$$

and so

$$T_\mu(\alpha x + \beta y) = \alpha T_\mu x + \beta T_\mu y.$$

(g) Let $x_1^* \in E^*$. Then considering the functions $f_1, f_2 : S \rightarrow E$, by $f_1(t) = T_t P x$ and $f_2(t) = P T_t x$ and applying them in (1.1), then we have

$$\mu_t \langle T_t P x, x_1^* \rangle = \langle f_1(t), x_1^* \rangle = \mu_t \langle (f_1)_\mu, x_1^* \rangle = \langle T_\mu P x, x_1^* \rangle$$

and

$$\mu_t \langle P T_t x, x_1^* \rangle = \langle f_2(t), x_1^* \rangle = \mu_t \langle (f_2)_\mu, x_1^* \rangle = \langle P T_\mu x, x_1^* \rangle,$$

for each $x \in C$. Since P commutes with $T_t \in \mathcal{S} = \{T_t : t \in S\}$ for each $s \in S$, we conclude that

$$\begin{aligned} \langle T_\mu P x, x_1^* \rangle &= \mu_t \langle T_t P x, x_1^* \rangle \\ &= \mu_t \langle P T_t x, x_1^* \rangle \\ &= \langle P T_\mu x, x_1^* \rangle, \end{aligned}$$

therefore $T_\mu P = P T_\mu$.

(h) Since X is a subspace of $l^\infty(S)$, $1 \in X$ and the mapping $t \rightarrow \langle T_t x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E$, hence, the mapping $t \rightarrow \langle T_t x - f, x^* \rangle$ is an element of X for each $x \in C$, $x^* \in E$ and $f \in \text{Fix}(\mathcal{S})$. For each $t \in S$, we have

$$\|T_t x - f\| \leq \|x - f\|,$$

for each $f \in \text{Fix}(\mathcal{S})$ and $x \in C$.

Suppose $f \in \text{Fix}(\mathcal{S})$ and $x_2^* \in J(T_\mu x - f)$, then from (1.1), we have

$$\begin{aligned} \|T_\mu x - f\|^2 &= \langle T_\mu x - f, x_2^* \rangle = \mu_t \langle T_t x - f, x_2^* \rangle \\ &\leq \sup_t \|T_t x - f\| \|T_\mu x - f\| \\ &\leq \|x - f\| \|T_\mu x - f\|. \end{aligned}$$

Then

$$\|T_\mu x - f\| \leq \|x - f\|,$$

and so T_μ is a $\text{Fix}(\mathcal{S})$ -quasi nonexpansive self mapping on C .

(i) Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as F -quasi nonexpansive self mappings on C that $F \subseteq \text{Fix}(\mathcal{S})$. Then for each $t \in S$, we have

$$\|T_t x - f\| \leq \|x - f\|,$$

for each $f \in F$ and $x \in C$. Suppose that $f \in F$, $x \in C$ and $x_2^* \in J(T_\mu x - f)$, then, as in the proof of (h), from (1.1), we have

$$\begin{aligned} \|T_\mu x - f\|^2 &= \langle T_\mu x - f, x_2^* \rangle = \mu_t \langle T_t x - f, x_2^* \rangle \\ &\leq \sup_t \|T_t x - f\| \|T_\mu x - f\| \\ &\leq \|x - f\| \|T_\mu x - f\|, \end{aligned}$$

thus

$$\|T_\mu x - f\| \leq \|x - f\|.$$

This means that T_μ is an F -quasi nonexpansive self mapping on C .

(j) Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as strongly F -quasi nonexpansive self mappings on C such that $F \subseteq \text{Fix}(\mathcal{S})$, then for each $t \in S$ we have

$$\|T_t x - f\| < \|x - f\|, \quad \forall (x, f) \in C \setminus F \times F.$$

Suppose that $f \in F$, $x \in C \setminus F$ and $x_2^* \in J(T_\mu x - f)$, then from (1.1), we have

$$\begin{aligned} \|T_\mu x - f\|^2 &= \langle T_\mu x - f, x_2^* \rangle = \mu_t \langle T_t x - f, x_2^* \rangle \\ &\leq \sup_t \|T_t x - f\| \|T_\mu x - f\| \\ &< \|x - f\| \|T_\mu x - f\|, \end{aligned}$$

then we have

$$\|T_\mu x - f\| < \|x - f\|,$$

therefore T_μ is a strongly F -quasi nonexpansive self mapping on C .

(k) Since $T_t^2 = T_t$ and the mapping $t \rightarrow \langle T_t x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E$, hence the mapping $t \rightarrow \langle T_t^2 x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E$. Suppose that $x \in C$ and $x_1^* \in E^*$, then from (1.1), we have

$$\begin{aligned} \langle T_\mu^2 x, x_1^* \rangle &= \mu_t \langle T_t^2 x, x_1^* \rangle \\ &= \mu_t \langle T_t x, x_1^* \rangle \\ &= \langle T_\mu x, x_1^* \rangle, \end{aligned}$$

hence $T_\mu^2 = T_\mu$.

(1) Since T_s is monotone for every $s \in S$, then we have $\langle T_s x - T_s y, x - y \rangle \geq 0$ for every $x, y \in H$ and $s \in S$. As in the proof of Theorem 1.4.1 in [13] we know that μ is positive i.e., $\langle \mu, f \rangle \geq 0$ for each $f \in X$ that $f \geq 0$. Then for each $x, y \in H$, from (1.1) we have

$$\langle T_\mu x - T_\mu y, x - y \rangle = \mu_t \langle T_t x - T_t y, x - y \rangle \geq 0,$$

then T_μ is a monotone self mapping on H . \square QED

Now we present some properties of T_μ in locally convex spaces.

Theorem 4. *Let S be a semigroup, E a locally convex space with predual locally convex space D , U a convex neighbourhood of 0 in D and p_U be the Minkowski functional. Let $f : S \rightarrow E$ be a function such that*

$$\langle x, f(t) \rangle \leq 1,$$

for all $t \in S$ and $x \in U$. Let X be a subspace of $l^\infty(S)$ such that the mapping $t \rightarrow \langle x, f(t) \rangle$ is an element of X , for each $x \in D$. Then, for any $\mu \in X^$, there exists a unique element $F_\mu \in E$ such that*

$$\langle x, F_\mu \rangle = \mu_t \langle x, f(t) \rangle,$$

for each $x \in D$. Furthermore, if $1 \in X$ and μ is a mean on X , then F_μ is contained in $\overline{\text{co}\{f(t) : t \in S\}}^{w^}$.*

Proof. We define F_μ by

$$\langle x, F_\mu \rangle = \mu_t \langle x, f(t) \rangle,$$

for each $x \in D$. Obviously, F_μ is linear in x . Moreover it follows from Proposition 3.8 in [9] that

$$|\langle x, F_\mu \rangle| = |\mu_t \langle x, f(t) \rangle| \leq \sup_t |\langle x, f(t) \rangle| \cdot \|\mu\| \leq p_U(x) \cdot \|\mu\|, \quad (1.2)$$

for all $x \in D$. Assume that (x_α) is a net in D that converges to x_0 . Then by (1.2) we have

$$|\langle x_\alpha, F_\mu \rangle - \langle x_0, F_\mu \rangle| = |\langle x_\alpha - x_0, F_\mu \rangle| \leq p_U(x_\alpha - x_0) \cdot \|\mu\|,$$

taking limit and using the continuity (see Theorem 3.7 in [9]) of p_U , we get F_μ is continuous on D and so $F_\mu \in E$.

Now, let $1 \in X$ and μ be a mean on X . Then, there exists a net $\{\mu_\alpha\}_I$ of finite means on X such that $\{\mu_\alpha\}_I$ converges to μ with the weak* topology on X^* . For each α , we may consider that

$$\mu_\alpha = \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} \delta_{t_{\alpha,i}},$$

such that $\lambda_{\alpha,i} \geq 0$ for each $i = 1, \dots, n_\alpha$ and $\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} = 1$. Therefore,

$$\langle x, F_{\mu_\alpha} \rangle = (\mu_\alpha)_t \langle x, f(t) \rangle = \langle x, \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} f(t_{\alpha,i}) \rangle,$$

for each $x \in D$ and $\alpha \in I$. Then we have

$$F_{\mu_\alpha} = \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} f(t_{\alpha,i}) \in \text{co}\{f(t) : t \in S\}.$$

Also

$$\langle x, F_{\mu_\alpha} \rangle = (\mu_\alpha)_t \langle x, f(t) \rangle \rightarrow \mu_t \langle x, f(t) \rangle = \langle x, F_\mu \rangle,$$

for each $x \in D$, therefore $\{F_{\mu_\alpha}\}$ converges to F_μ in the weak* topology and

$$F_\mu \in \overline{\text{co}\{f(t) : t \in S\}}^{w*},$$

we can write F_μ by $\int f(t) d\mu(t)$. \square

In the next we show that T_μ inherits some properties of the representation \mathcal{S} in locally convex spaces.

Theorem 5. *Let S be a semigroup, C a closed convex subset of the locally convex space E . Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, \mathcal{B} a base at 0 for the topology E which consists of convex and balanced sets. Let $Q = \{q_V : V \in \mathcal{B}\}$ where q_V is the associated Minkowski functional with V . Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as Q - G -nonexpansive mappings from C into itself and X be a subspace of $B(S)$ with $1 \in X$ and μ be a mean on X such that the mapping $t \rightarrow \langle T_t x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. If we write $T_\mu x$ instead of $\int T_t x d\mu(t)$, then the following facts hold:*

- (i) T_μ is a Q - G -nonexpansive mapping from C into C .
- (ii) $T_\mu x = x$ for each $x \in \text{Fix}(\mathcal{S})$.
- (iii) If the dual of E is a locally convex space with predual locally convex space D and C a w^* -closed convex subset of E and U a convex neighbourhood of 0 in D and p_U is the associated Minkowski functional. Let the mapping $t \rightarrow \langle z, T_t x \rangle$ be an element of X for each $x \in C$ and $z \in D$, then

$$T_\mu x \in \overline{\text{co}\{T_t x : t \in S\}}^{w*}.$$

(iv) if X is r_s -invariant for each $s \in S$ and μ is right invariant, then

$$T_\mu T_t = T_\mu,$$

for each $t \in S$.

(v) let $a \in E$ be a Q - G -attractive point of \mathcal{S} and the mapping $t \rightarrow \langle a - T_t x, x^* \rangle$ be an element of X for each $x \in C$ and $x^* \in E$, then a is a Q - G -attractive point of T_μ .

Proof. (i) Let $x, y \in C$ and $V \in \mathcal{B}$. By Proposition 3.33 in [9], the topology on E induced by Q is the original topology on E . By Theorem 3.7 in [9], q_V is a continuous seminorm and from Theorem 1.36 in [8], q_V is a nonzero seminorm because if $x \notin V$ then $q_V(x) \geq 1$, hence from Theorem 2, there exists a functional $x_V^* \in X^*$ such that

$$q_V(T_\mu x - T_\mu y) = \langle T_\mu x - T_\mu y, x_V^* \rangle,$$

and $q_V^*(x_V^*) = 1$. Also from Theorem 3.7 in [9], $q_V(z) \leq 1$ for each $z \in V$.

We conclude that $\langle z, x_V^* \rangle \leq 1$ for all $z \in V$. Therefore from Theorem 3.8 in [9], $\langle z, x_V^* \rangle \leq q_V(z)$ for all $z \in E$. Hence for each $t \in S$, $x, y \in C$ that $(x, y) \in E(G)$ and $x^* \in E^*$, from (1.1), we have

$$\begin{aligned} q_V(T_\mu x - T_\mu y) &= \langle T_\mu x - T_\mu y, x_V^* \rangle = \mu_t \langle T_t x - T_t y, x_V^* \rangle \\ &\leq \|\mu\| \sup_t |\langle T_t x - T_t y, x_V^* \rangle| \\ &\leq \sup_t q_V(T_t x - T_t y) \\ &\leq q_V(x - y), \end{aligned}$$

then we have

$$q_V(T_\mu x - T_\mu y) \leq q_V(x - y),$$

for all $V \in \mathcal{B}$.

(ii) Let $x \in \text{Fix}(\mathcal{S})$ and $x^* \in E^*$. Then we have

$$\langle T_\mu x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \mu_t \langle x, x^* \rangle = \langle x, x^* \rangle.$$

(iii) The assertion follows from Theorem 4.

(iv) This part obtains from the following equalities:

$$\langle T_\mu(T_s x), x^* \rangle = \mu_t \langle T_{ts} x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \langle T_\mu x, x^* \rangle.$$

(v) Let $x \in C$ and $V \in \mathcal{B}$. From Theorem 2, there exists a linear functional $x_V^* \in X^*$ such that

$$q_V(a - T_\mu x) = \langle a - T_\mu x, x_V^* \rangle,$$

and $q_V^*(x_V^*) = 1$. It follows from [9, Theorem 3.7] that $q_V(z) \leq 1$ and $\langle z, x_V^* \rangle \leq 1$, for each $z \in V$. Therefore Theorem 3.8 in [9] implies

$$\langle z, x_V^* \rangle \leq q_V(z),$$

for each $z \in E$. Then by applying (1.1) and for each $t \in S$ and $x, y \in C$ that $(x, y) \in E(G)$ and $x^* \in E^*$, we have

$$\begin{aligned} q_V(a - T_\mu x) &= \langle a - T_\mu x, x_V^* \rangle = \mu_t \langle a - T_t x, x_V^* \rangle \\ &\leq \|\mu\| \sup_t |\langle a - T_t x, x_V^* \rangle| \\ &\leq \sup_t q_V(a - T_t x) \\ &\leq q_V(a - x), \end{aligned}$$

and

$$q_V(a - T_\mu x) \leq q_V(a - x),$$

for all $V \in \mathcal{B}$.

\square

2 Conclusion

In this paper, we prove that some properties of the mapping in the representation $\mathcal{S} = \{T_s : s \in S\}$ can be transferred to the mapping T_μ introduced by a mean on a subspace of $B(S)$, for example nonexpansiveness, quasi-nonexpansiveness, strongly quasi-nonexpansiveness, monotonicity, retraction property and another properties in Banach spaces, and Q - G -nonexpansiveness using a directed graph in locally convex spaces.

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